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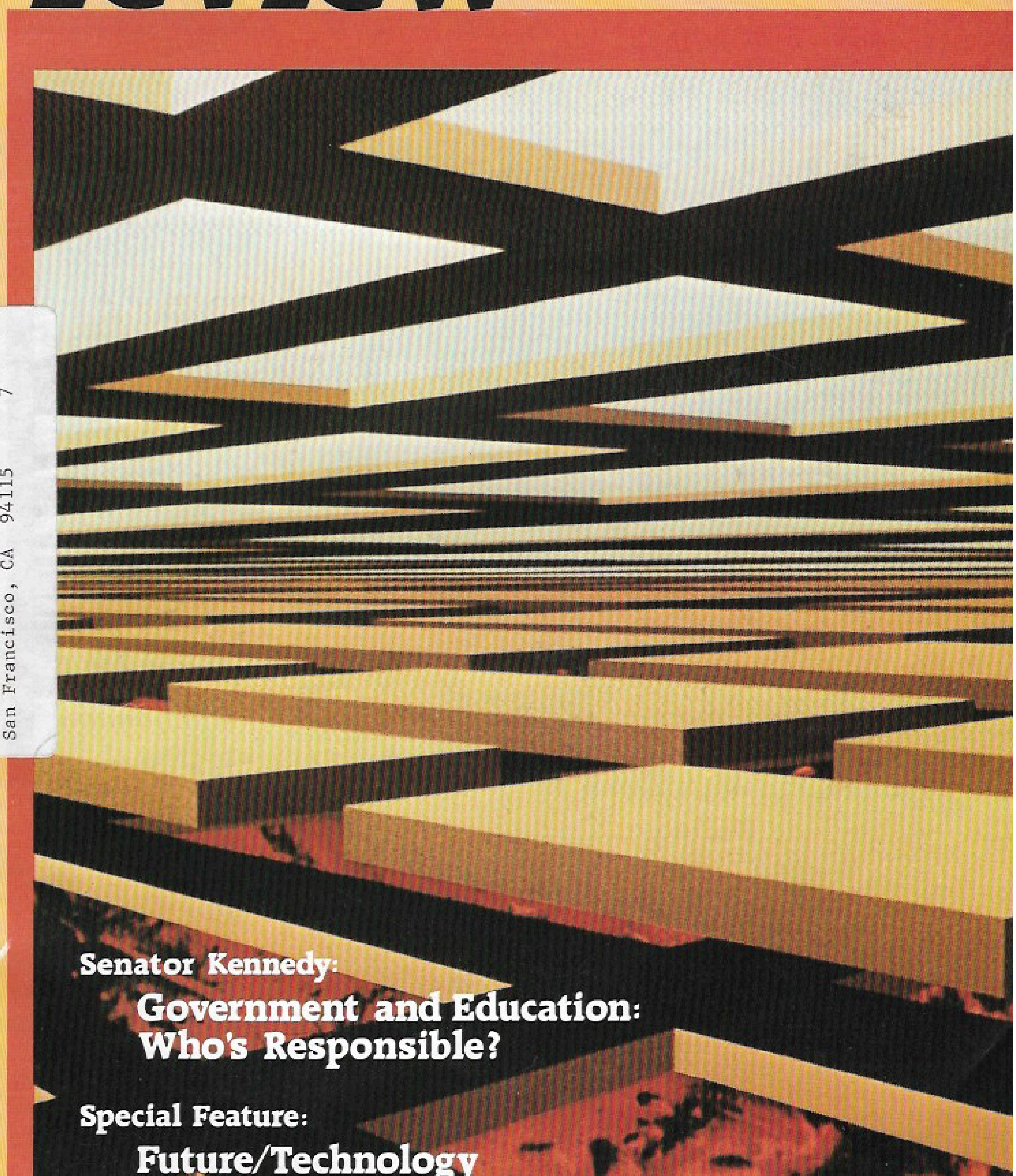

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Teaching Mathematics As Ideas

by Craig McGarvey

By making historical connections, by bringing great mathematicians and their ideas to life, we elevate the endeavors of our students.

The study of mathematics is apt to commence in disappointment. . . . The unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception.

—Alfred North Whitehead
An Introduction to Mathematics

Why is the study of mathematics so often disappointing not only to students but to their teachers? "The reason for the failure of the science to live up to its reputation," Whitehead says, "is that its fundamental ideas are not explained to the student disentangled from . . . technical procedure." We are pedants, Whitehead charges—and here, of course, he is indicting the English school curriculum of the early years of our century—we are pedants in our refusal to rise above the minutiae of mathematical technique to teach mathematical ideas.

In our own country, in the years since the publication of *An Introduction to Mathematics*, the school curriculum has experienced great movements for change. In the 1950s, the move was away from mathematics as skill and technique, and toward mathematics as symbolic language. Mathematics as skill made a comeback in the last decade and is still a powerful force. Most recently, the problem solving thrust seems to be arguing for mathematics as application. We have given much collective thought and energy to our math curricula. Yet I would contend that, in our continuing refusal to teach *about* mathematics while we teach mathematics, we have condemned our students to a study that still commences and, for many, ends in disappointment. I wish to propose an alternative approach to the teaching of mathematics that is fundamentally different in emphasis.

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The History of Ideas

The year's course of study I shall describe here follows first-year algebra and geometry; it is taught as the last in the sequence of mathematics courses required for graduation from our college preparatory high school. With the exception of a few topics normally covered in geometry, the content of the course is that of a traditional class in advanced algebra and trigonometry, and most of the homework problems are drawn from standard textbooks. The approach, however, is very different from the standard course.

Its central theme is the historical development of mathematical ideas. From ancient times to

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the modern era, our study follows the major evolutionary lines of mathematical thought. Each trigonometric topic we cover, each algebraic skill we practice is motivated and informed by this historical approach.

History personifies ideas, and throughout the course we take our lead from the great mathematicians themselves. Problems distilled from their books introduce our studies. Consideration of their lives and work also helps to make explicit the association between the mathematics of the modern curriculum and larger mathematical ideas.

The intent of such an approach is neither to de-emphasize the teaching of technique nor to dilute mathematical concepts. On the contrary: embedding the study of mathematics within a larger cultural and philosophical context reinforces skills learning and gives Whitehead's "unfortunate learner" a framework on which to build a knowledge of those "masses of details."

Beginning in Babylonia

Our study begins early in the historical record with a comparison of the mathematics of ancient Babylonia and Egypt and the mathematics of Greece. Master empiricists and inductive reasoners, the Babylonians were the great algebraists of the ancient world. The Greeks gave mathematics the priceless invention of deduction, but with their relentless geometric view they were hesitant before Babylonian algebra.

Comparing these two approaches in the classroom can help students to see the distinction between the inductive, algorithmic mode of mathematical reasoning and the deductive, axiomatic mode. Further, the comparison dramatizes for students the division between algebra and geometry which began with the Greeks and lasted until the invention of analytic geometry in the 17th century.

The mathematicians placed in the spotlight are Thales and Pythagoras, who helped give birth to the new method; Euclid, who synthesized early Greek mathematics and presented the first model of a deductive system; and Archimedes of the Alexandrian School. Our specific study includes the circle and line relationships from geometry and the areas and volumes of circular figures.

The second unit focuses on the relationship between mathematics and science in the ancient world. We study the first exact science, astronomy (the first mathematicians *were* the astronomers), and the mathematics of indirect measurement which emerged from it, trigonometry.

Eratosthenes, Aristarchus of Samos, and Hipparchus introduce us to the concepts of right triangle trigonometry. Hipparchus and, in the *Almagest*, Claudius Ptolemy needed the sum and difference identities to generate their trigonometric tables. We follow their lead and develop these identities for angle measures between 0° and 90°. Ptolemy also used the laws of sines and cosines in his *Almagest* calculations, and we therefore take this opportunity to develop them, once again for restricted angle measure.

Steps to Symbolic Algebra

The next unit of study covers a broad sweep of historical time: the almost 4000 years during which mathematicians slowly developed symbolic algebra. Our theme is the importance of symbolism, the ease and facility it imparts to mathematical thought and communication, and the help it lends in solving problems.

It is interesting to note that algebra problems just like those found in 20th century school texts appear on some of the earliest Babylonian tablets. In the third century B.C. the Babylonians knew the equivalent of the method of completing the square to solve the equivalent of quadratic equations. What they did not know was symbolic algebra. Their method appears as an algorithm, a step-by-step recipe of written rules which occupied the abilities of the best mathematicians of their age. We, who are not the best mathematicians of our own age, have an easier time with quadratics because we have learned symbolic algebra. In the course this point is made by developing the method of completing the square symbolically, then using it in the general case to develop the quadratic formula.

We go on to bridge the relatively unproductive gap from the ancient world, through the Hindus (who took a step toward symbolic algebra), and the Moslems (who took a step back), to 12th century Europe, with a study of simultaneous equation problems drawn from the historical record. We then follow five centuries of European development of algebra, motivating in the process a study of rational exponents.

Mathematics is not only skill, not only symbolic language, not only application. Surely it is all of these and more.

Blossoming of Genius

The next two units focus on the 17th century, the century of mathematical genius. First, Pierre de Fermat and René Descartes introduce us to their great co-invention, analytic geometry. Like the mathematicians of the time, we look back to the heritage of the classical world, reformulating into algebraic equations the Greek locus definitions of the conic sections.

Then, with the help of Johannes Kepler, Galileo Galilei, and Isaac Newton, we study the scientific revolution, making note of the wedding of induction and deduction in methodology, of the banishment of mathematics from the world of argument to that of description. We apply our earlier parabola work to the description of projectile motion, learn vector arithmetic, and introduce the concept of function.

The 18th century saw the blossoming of mathematics: the exploitation of Newton's calculus, the movement toward abstraction and generalization, the growing distinction between pure and applied mathematics, the rising call for rigor in mathematical development. In the spirit of the age we study the function concept—in particular, the circular, exponential, and logarithmic functions—both in the abstract and in application to the natural world. We receive help from Daniel Bernoulli, Leonard Euler, Joseph Lagrange, Pierre Simon Laplace, and Joseph Fourier.

To capture the spirit of the 19th century—the search for structure and logical foundations, the axiomatization of arithmetic and algebra—we study the extension of the number system to complex numbers. Our guides are Carl Friedrich Gauss, William Rowan Hamilton, and David Hilbert.

The course closes with two units on the mathematical inventions which have helped to change the modern world: the development of statistical science and probability theory, and the development of computer science.

The Scientific Spirit

The medium of our study is often the message. The unit on projectile motion, for example, is launched with readings from Galileo's *Dialogues Concerning Two New Sciences.* We begin with Galileo's experimentally determined observation that falling bodies are uniformly accelerated; that is, they acquire equal increments of velocity in equal intervals of elapsed time. Using the concept of average velocity, working from the case of a projectile falling from rest to that of a projectile fired at a given velocity and angle of elevation, we develop the formulas for velocity and position as a function of time, and we solve problems with them.

In so doing we are working within the spirit of modern science, a spirit which Galileo articulated quite clearly in the *Dialogues.* We control some of the variables (resistance due to air is neglected, for example). We apply mathematics as a descriptive language of nature. We check our mathematical deductions against inductive evidence often in the form of thought experiments from our own experience (a book dropped in a car traveling at a constant speed falls on the lap and does not fly back into the stomach).

Using analytic geometry in the projectile unit is deliberately anachronistic, for Galileo did not know of it. However, the larger point is that the course has no interest in teaching the mathematics of antiquity. The trigonometric tables in Ptolemy's *Almagest* are tables of chord lengths versus arc measures in circles of effectively unit radius. Chords are the ancestors of sines, and it is modern trigonometry which we study.

Recurring Patterns

In addition to the historical themes, the course interweaves several topical and conceptual themes. Quadratics recur, as one might expect. The circle resurfaces throughout. We study its geometry and its analytic geometry; we consider its relationships to trigonometry and the conic sections. The circle also occupies a philosophical position. The ancient Greeks chose uniform circular motion as the fundamental motion of the universe; however, 17th century mathematicians cast aside the circle and, in the concept of inertia, chose the straight line and uniform linear motion.

The evolution of physics and of cosmological theories motivates much of our work throughout, and the relationship between science and mathematics, between pure and applied mathematics, is central to the course. The tendency of scientific inquiry to motivate mathematical invention, for instance, is displayed in the interplay between

The thematic approach creates meaning which will stay with students long after skills have been lost and topics forgotten.

Greek astronomy and trigonometry.

When we have generated the sum and difference identities for acute angle measures, recognizing in the process their implications for expanded definitions of the trigonometric ratios, we make note of the tendency of mathematics to take flight into abstraction. Later, from the perspective of the 18th and 19th centuries, we watch trigonometry touch ground again in the application of the circular functions to harmonic motion and electromagnetic theory.

Throughout the course, relationships between inductive and deductive modes of reasoning also recur, as do relationships between algebra and geometry.

Why Bother?

The question legitimately arises, why bother? Why teach historical background when there is already scarcely enough time in a school year to cover essential mathematical topics? If we agree with Whitehead's statement that powerful mathematical ideas often get bogged down in the study of technique, we should easily see the need to set those essential topics in the perspective of their underlying ideas.

Mathematics is not only skill, not only symbolic language, not only application. Surely it is all of these; it is creative art form; it is more. The historical study outlined here is an inclusive rather than an exclusive approach to mathematics, and at its heart lies a belief in mathematics as idea. The ideas of mathematics have evolved through the centuries, this course says, and they are alive in the topics of the secondary curriculum.

The task of mathematics education is to impart technical facility, yes, to give practice with orderly thinking and the solution to problems, to teach symbolic language. But it is also to help students create personal meaning in their understanding of mathematics. The thematic, evolutionary approach, in its association of specific mathematics with larger ideas, creates meaning which I believe will stay with my students long after skills have been lost and topics forgotten.

The historical approach facilitates associations between mathematics and other areas of human study. As often as not, my students themselves make these connections. A young philosopher gives the class a different perspective on Descartes. A student of art history or of English literature notices the similar concerns within an era of separate disciplinary movements.

There is a further humanizing element in the approach. I am not prepared to defend as either a theory of learning or of history the notion that "ontogeny recapitulates phylogeny." Yet there are parallels of pedagogical value between the stages of individual learning and the historical development of mathematics. When one of my students says to me that word problems are easy, the only difficult part is getting them translated into algebra, I answer that the translation should be difficult. Mankind took almost 4000 years to make the step.

I have described this approach as a total year's course for upper level math students, but in fact its principles are accessible to all teachers of mathematics. Although classroom-ready materials are presently scarce, excellent sources are readily available for the interested teacher to select and combine: chronological surveys, thematic treatments, biographies, English translations of the great works, English collections of selected passages from primary sources.

A math teacher at any grade level might begin by presenting a single problem from its historical perspective. A single chapter from the textbook or a unit from the syllabus could be redesigned. In my experience, this year's experimental lesson plan can easily become next year's two-week presentation. In just this way, step by step, unit by unit, my course has developed.

The efforts at such development return great rewards. The frustrations, insights, failures, and triumphs of each of our students are part of the historical fabric of intellectual growth. By making historical connections, by bringing mathematicians and their ideas to life inside the classroom, we elevate the endeavors of our students. In so doing we endow their work with meaning. By teaching mathematical ideas while we teach mathematics, we educate.

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